Games and perfect independent subsets of the generalized Baire space

Dorottya Sziráki

For an uncountable cardinal $\kappa = \kappa^{<\kappa}$, the generalized Baire space ${}^\kappa\kappa$ is the space of functions $\kappa \to \kappa$ equipped with the bounded topology. Jouko Väänänen introduced a notion corresponding to perfectness for ${}^\kappa\kappa$, based on a game $G_\kappa(X)$ of length κ played on subsets $X \subseteq {}^\kappa\kappa$. Consider the following dichotomy:

> for all subsets $X \subseteq {}^\kappa\kappa$, either $|X| \leq \kappa$ or player II has a winning strategy in $G_\kappa(X)$ (i.e., there exists $Y \subseteq X$ whose closure $\overline{Y}$ is κ-perfect).

In joint work with Philipp Schlicht, we show that the existence of a weakly compact cardinal $\lambda > \kappa$ implies the consistency of this dichotomy. As a corollary, it also implies the consistency of a κ-perfect set theorem about the possible sizes of $\mathcal{R}$-independent subsets of any $X \subseteq {}^\kappa\kappa$ with respect to collections $\mathcal{R}$ of κ many $\mathbf{\Sigma}^0_2(\kappa)$ finitary relations on X. (A subset $Y \subseteq X$ is called $\mathcal{R}$-independent if for all n-ary $R \in \mathcal{R}$ and pairwise distinct $y_1, \ldots, y_n \in Y$ we have $(y_1, \ldots, y_n) \notin R$.) It has been known that the consistency strength of either of the above mentioned dichotomies lies between the existence of an inaccessible cardinal above κ and the existence of a measurable cardinal above κ.

By considering a modification of $G_T(X)$ of the above game, also introduced by Jouko Väänänen, which allow trees T to play a role analogous to that of Cantor-Bendixson ranks for the generalized Baire space ${}^\kappa\kappa$, I also obtain a version of the above κ-perfect set theorem for $\mathcal{R}$-independent sets which already holds assuming only $\diamondsuit_\kappa$ or the inaccessibility of κ.